Derived Category in Representation of Groups

William Wong

City University

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Notations and Aims

- $\mathbf{R}$ is a (Noetherian) ring. All R-modules are left modules.
- M is an $\mathbf{R}$-module, an object of $\mathbf{R}\text{-}\mathrm{Mod}$.
- Chain complex might means cochain complex - they are the same anyway.
- Tensor product of two left kG-modules: Tensor as vector space equipped with diagonal G-action.

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- Tensor product of two left kG-modules: Tensor as vector space equipped with diagonal G-action.

At the end we hope you have an idea of:

- Homotopy category and Derived category;
- Tensor products on derived category;
- Verdier quotient: Quotient triangulated category;
- Rickard's Theorem: For modules of group algebras, stable category is a triangulated quotient of derived category.

Homotopy Category

Definition 1 (Homotopy category)

The homotopy category of $\mathbf{R}\text{-}\mathrm{Mod}$, denoted $K(\mathbf{R})$ (we omit $\text{-}\mathrm{Mod}$), has
Object: Chain complexes of objects of $\mathbf{R}\text{-}\mathrm{Mod}$
Morphism: Chain map modulo chain homotopy.

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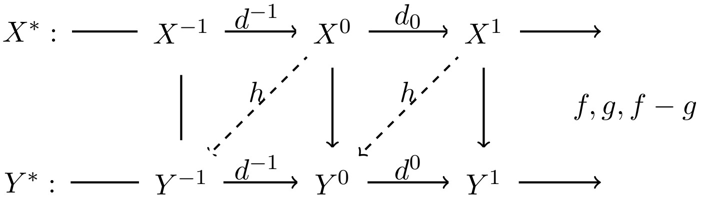
Reminder: Two maps $f, g : X_* \to Y_*$ are chain homotopic if there exist a degree 1 map h such that $f - g = d \circ h + h \circ d$.



About Our Objects: Chain complexes

A chain complex of $\mathbf{R}$-modules, X^* is

$$X^* : \cdots \to X^{-1} \xrightarrow{d^{-1}} X^0 \xrightarrow{d^0} X^1 \to \dots .$$

- Each X^i is a $\mathbf{R}$-module, each d^i is a module map.
- $d^{i+1} \circ d^i = 0$, or simply $d \circ d = 0$.

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There is a full embedding of $\mathbf{R}\text{-Mod}$ to $K(\mathbf{R})$:
An object $M \in \mathbf{R}\text{-Mod}$ is regarded as chain complex

$$... \to 0 \to M \to 0 \to ...$$

where M is in the zeroth position.
We shall be using this embedding from now on. That is, we regard an object in $\mathbf{R}\text{-Mod}$ as object in $K(\mathbf{R})$ by this embedding.
Sometimes we write X in place of $X^\bullet$.

Projective Resolution

Let M be a $\mathbf{R}$-module. A projective resolution of M, P_M is a chain complex of $\mathbf{R}$-projective modules

$$\cdots \to P_2 \to P_1 \to P_0 \to 0$$

such that each P_i is projective $\mathbf{R}$-module and $H_n(P_M) = M$ when $n = 0$, zero otherwise.

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1. You can also have projective resolution to a chain complex of $\mathbf{R}$-modules M^*.
2. Existence can be guaranteed, uniqueness up to homotopy equivalence.
3. Dual construction of injective resolution.

Quasi-isomorphism

Two Chain complexes X and Y are quasi-isomorphic if there is a chain map f from X to Y such that f^* is an isomorphism of their homology groups.

e.g. In $K(\mathbf{R})$, an $\mathbf{R}$-module M (as chain complex) and its projective resolution P_M is quasi-isomorphic.

$$\begin{array}{cccccccccc} P_M : & ... \to & P_M^{-1} & \to & P_M^0 & \to & 0 & \to ... \\ & & & & \downarrow & & & \\ M : & ... \to & 0 & \to & M & \to & 0 & \to ... \end{array}$$

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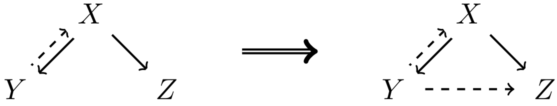
Localise: Add formal inverse (add f^{-1} for q.i. f) $\Rightarrow$ Derived category.

e.g. 'Rooftops'



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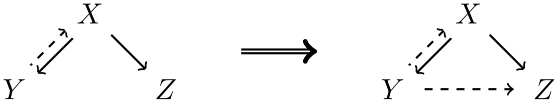

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Note: Homotopy equivalences are examples of quasi-isomorphisms.

Derived Category

Definition 2 (Derived category)

The derived category of $\mathbf{R}\text{-}\mathrm{Mod}$, denoted $D(\mathbf{R})$, has
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It is possible to define derived category directly without introducing homotopy category - but it is more complex than you might think.
Technical: We done most of the calculation in $D(\mathbf{R})$ using projective resolution. Which practically is working in $K(\mathbf{R})$.

Remark

$K(\mathbf{R})$ and $D(\mathbf{R})$ is a triangulated category. Shift is the suspension Σ; Mapping cones construct standard triangles. - see next page.

Operations On $K(\mathbf{R})$ and $D(\mathbf{R})$

- Shift functor (denoted [1]): Moving chain complex 1 space left

$$X[1] = ... \xrightarrow{-d^{-1}} X^0 \xrightarrow{-d^0} \underline{X^1} \xrightarrow{-d^1} X^2 \xrightarrow{-d^2} ...$$

where d is negated and the underline term is the zeroth chain.

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- Mapping cone: For $X \xrightarrow{f} Y$, the mapping cone of f is a chain complex

$$Cone(f) \text{ with terms } X[1] \oplus Y \text{ and differential } \begin{pmatrix} d_{X[1]} & 0 \\ f[1] & d_Y \end{pmatrix}$$

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- Hom operation

$$(\mathrm{Hom}_{D(\mathcal{A})}(X,Y))^i = \Pi_j \mathrm{Hom}_{\mathcal{A}}(X^j, Y^{j+i})$$

with differential $(df)(v) = d(f(v)) - (-1)^{|f|} f(d(v))$.
We do not need this immediately.

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- Tensor product (on chain complexes)

$$(X \otimes Y)^i = \bigoplus_{j+k=i} X^j \otimes Y^k$$

with differential $d(a \otimes b) = da \otimes b + (-1)^{|a|} a \otimes db$.

Our Tensor Product on $D(kG)$

Recall we can define tensor product for two left kG-modules: (We can't do it for any ring $\mathbf{R}$ except commutative ones.)
Let M, N be two left kG-modules with basis m_i, n_j Define $M \otimes N$ to be the vector space with basis $m_i \otimes n_j$ equipped with diagonal G-action: For $g \in G$,

$$g.(m \otimes n) = g.m \otimes g.n.$$

Note the tensor product $- \otimes N$ is exact, so quasi-isomorphism is preserved. Extend this definition the chain complex of left kG-modules we get a tensor product structure well-defined on $D(kG)$ - makes $D(kG)$ a symmetric monoidal (tensor) category in categorical terms. This tensor product also preserves triangles, so $D(kG)$ is a tensor triangulated category.

Quotient Categories for Triangulated Categories

Definition 3 (Thick Triangulated Subcategory)

A *triangulated subcategory* is a full (triangulated) subcategory $\mathcal{S} \subset \mathcal{T}$ of a triangulated category such that

1. Contains the zero object (or non-empty).
2. It is closed under suspension (shift in $K(\mathbf{R}), D(\mathbf{R})$; Heller translate in $kG\text{-}\underline{\mathrm{Mod}}$).
3. If two terms of a triangle belong to $\mathcal{S}$ so is the third (e.g. cones).

Such subcategory is *thick (épaisse)* if direct summands of an element in $\mathcal{S}$ is in $\mathcal{S}$.

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One might regard thick subcategory as subcategory for quotients. (Normal subgroups; ideals.)
The quotient category (denoted $\mathcal{T}/\mathcal{S}$ from above definition) is triangulated. This is the Verdier quotient of triangulated categories.

Verdier Quotient: $K(\mathbf{R})$ to $D(\mathbf{R})$

Example 4

Acyclic complexes of $\mathbf{R}$-mod (complexes with zero homology) forms a thick subcategory of $K(\mathbf{R})$.

The Verdier quotient effectively treats the object in $\mathcal{S}$ as zero. Hence, consider these two triangles in $K(\mathbf{R})$,

$$\begin{array}{ccccccc} P_M & \xrightarrow{f} & M & \rightarrow & Cone(f) & \rightsquigarrow \\ \downarrow & & \downarrow & & \downarrow & \\ M & \xrightarrow{id} & M & \rightarrow & 0 & \rightsquigarrow \end{array}$$

we forced f to have an inverse because $Cone(f)$ is acyclic (and identity definitely have inverse), thus effectively inverting quasi-isomorphisms.

Verdier Quotient: $D^b(kG)/D^{per}(kG) \cong kG\text{-}\underline{\text{mod}}$

Our last job of this talk will see stable module category kG-$\underline{\text{mod}}$ being a quotient category of $D^b(kG)$.

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First, we are in bounded derived category with objects having finitely-generated homology in only finitely many degrees. This does not change anything we have already discussed. Second, we restrict ourselves to kG-mod (self-injective algebra) with finitely generated modules so that kG-$\underline{\text{mod}}$ is triangulated (not true in general).

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Definition 5

A perfect complex of $D^b(kG)$ is a chain complex quasi-isomorphic to a bounded complex of finite projective kG-modules.

It is easy to check all perfect complexes forms a thick subcategory $D^{per}(kG)$ in $D^b(kG)$.

Theorem 6 (Rickard's Theorem)

The Verdier quotient $D^b(kG)/D^{per}(kG)$ is equivalent to kG-$\underline{\text{mod}}$ as triangulated categories.

We give a very brief sketch of proof here

1. Consider an additive functor $F' : kG\text{-Mod} \to D^b(kG)/D^{per}(kG)$. All kG-projective modules are being treated as zero, since the chain of them concentrated in degree zero is a perfect complex. Thus F' factors through to $F : kG\text{-}\underline{\text{Mod}} \to D^b(kG)/D^{per}(kG)$.
2. Exactness of F using the pushout diagram on kG-modules.
3. Fullness and faithfulness of F by properties of kG-modules.
4. Every object X in $D^b(kG)/D^{per}(kG)$ is isomorphic to $F(M)$ for some module M. Done by truncating projective resolution of X and using cone.